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Fractals without anomalous diffusion

Raffaella Burioni*
Dipartimento di Fisica, Università di Roma "La Sapienza," Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Piazzale Aldo Moro 1, 00185 Roma, Italy

Davide Cassi†
Dipartimento di Fisica, Università di Parma, Istituto Nazionale di Fisica della Materia–Consorzio Interuniversitario di Struttura della Materia, Unità di Parma, Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Parma, Viale delle Scienze, 43100 Parma Italy
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By an alternative analytical technique we solve the random-walk problem on a particular class of fractal trees, calculating the exact value of their spectral dimension. The result shows that for all these structures the spectral dimension and the fractal dimension are equal, providing an example of fractal structures with nonanomalous diffusion as well as an example of fractals with a noninteger spectral dimension greater than 2.

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Fractal dynamics is one of the most important and studied subjects in disordered systems. Since the seminal paper of Alexander and Orbach [1] where the spectral dimension $\tilde{d}$ [2] was introduced starting from anomalous diffusion on fractals, a lot of work has been done in order to clarify its role in determining various relevant physical properties of these structures. The first definition of $\tilde{d}$ was based on the assumption of an anomalous power-law behavior at long times t for the mean-square displacement $\langle r^2 \rangle$ of a diffusing particle, according to

$$\langle r^2(t) \rangle \sim t^{2/(2+\delta)} , \tag{1}$$

where δ is a non-negative number characterizing the diffusion anomaly. Calling d_f the fractal dimension of the structure, the spectral dimension is then given by

$$\tilde{d} = \frac{2d_f}{2+\delta} . \tag{2}$$

This definition implies that the probability $P_0(t)$ for a random walker of returning to the starting site after t steps has the asymptotic behavior [1]

$$P_0(t) \sim t^{-\tilde{d}/2} \tag{3}$$

and this law can be used for an equivalent definition of $\tilde{d}$. Notice that from (2) it follows that $\tilde{d} \le d_f$. This inequality is fulfilled for d-dimensional Euclidean structures where $\tilde{d} = d_f = d$. In this case we say that the system has *dimensional degeneracy*, $\delta = 0$ and the exponent of t in (1) is 1: in other words, the diffusion is nonanomalous. Spectral dimension controls not only the diffusion on fractals, but also the vibrational spectrum, the critical behavior of the Gaussian model, and in general all properties related to harmonic equations [3]. Moreover, its

*Electronic address: BURIONI@ROMA1.INFN.IT, VAXROM::BURIONI
†Electronic address: CASSI@PR.INFN.IT, 37993::CASSI

value is crucial in determining the possibility of phase transitions on these structures: indeed it is impossible to have spontaneous breaking of a continuous symmetry when $\tilde{d} \leq 2$ [4]. It is interesting to notice that up to now, both in experimental and theoretical literature, there are no examples of fractals with $\tilde{d} > 2$, so that it has been impossible to explore the possibility of a phase transition in statistical models defined on this kind of structure. Moreover, for all known fractals δ is greater than zero. Therefore, it is commonly believed that diffusion on fractals is always anomalous and that dimensional degeneracy appears only in the Euclidean case.

In this Rapid Communication we calculate analytically the spectral dimension of a particular class of fractal trees, called NT_D (nice trees of dimension D, defined as trees whose branches are splitting in r every time the distance from the origin is doubled, where r is an integer greater than 1; the relation between r and D is given below) in mathematical literature [5], finding that its value coincides with fractal dimension. This provides an example of fractals with dimensional degeneracy and without anomalous diffusion. In addition this equality together with the existence of NT_D with a fractal dimension greater than 2 gives also an example of structures with $\tilde{d} > 2$.

NT_D can be recursively constructed in the following way [5]: take an origin point O and connect it to another point A by a link (branch of length 1); then attach to A r branches of length 2; at the end of each of these branches attach r branches of length 4 and so on attaching to each branch of length 2^n r branches of length 2^{n+1} up to infinity (see Fig. 1). It is possible [8] to choose a natural embedding of NT_D in Euclidean space in such a way that their fractal dimension d_f coincides with its connectivity dimension D [7] given by

$$D = 1 + \ln r / \ln 2 \ . \tag{4}$$

We introduce discrete time random walks on these structures defining the probability for a walker on site i at time t of jumping on a nearest neighbor at time $t+1$ to be $1/z_i$, where z_i is the coordination number of i. In the following we will calculate the probability $P_0(t)$ for a walker started from O of returning to O after t steps. Since the value of $\tilde{d}$ is independent of the starting site [3], from the asymptotic behavior of $P_0(t)$ at large t we will obtain the spectral dimension of NT_D. We give now the main steps of this calculation made by an alternative analytical technique without using any renormalization method, as is instead usual in fractal physics. Mathematical details as well as a complete description of the technique will be given in a forthcoming specialistic paper [6]. Let us consider one of the r subtrees ST_A starting from A and call $P_A(t)$ the probability of returning to A for a random walker on it with jumping probabilities defined as above. Then define $P'_A(t) = P_A(2t)$. Due to the particular geometry of NT_D, $P'_A(t)$ is equal to the probability $P_0^*(t)$ of returning to O for a random walk on the original tree with modified jumping probabilities equal to $1/2z_i$ and with a probability of staying in the site equal to $\frac{1}{2}$. If we now introduce the generating function $\tilde{P}_0(\lambda) = \sum_{t=0}^{\infty} P_0(t)\lambda^t$ and the analogous ones for the other probabilities, from the above considerations it can be shown that

$$\tilde{P}_A(\lambda) = \tilde{P}_0^*(\lambda^2) \tag{5}$$

and

$$\tilde{P}_0^*(\lambda) = \frac{2}{2-\lambda}\tilde{P}_0\left[\frac{\lambda}{2-\lambda}\right] . \tag{6}$$

Moreover, considering that the NT_D can be obtained joining in A the r ST_A and the link OA, by a method already used in [8], we easily get

$$\tilde{P}_0(\lambda) = \frac{\tilde{P}_A(\lambda) + r}{\tilde{P}_A(\lambda)(1-\lambda^2) + r} . \tag{7}$$

Now from (5), (6), and (7) we get the exact functional equation for $\tilde{P}_0(\lambda)$

$$\tilde{P}_0(\lambda) = \frac{[2/(2-\lambda^2)]\tilde{P}_0[\lambda^2/(2-\lambda^2)] + r}{(2-2\lambda^2)/(2-\lambda^2)\tilde{P}_0[\lambda^2/(2-\lambda^2)] + r} . \tag{8}$$

Since random walks generating functions on graphs with finite connectivity dimension always have convergence radius equal to 1 [9], we can introduce the asymptotic development of the singular part of $\tilde{P}_0(\lambda)$ for $\lambda \to 1^-$, obtaining the solution [8]

$$\mathcal{S}[\tilde{P}_0(\lambda)] \sim \begin{cases} (1-\lambda)^{1/2(\ln r/\ln 2 - 1)} & \text{for } r \neq 2^n \\ (1-\lambda)^{1/2(\ln r/\ln 2 - 1)}\ln(1-\lambda) & \text{for } r = 2^n \ . \end{cases} \tag{9}$$

Now from standard Tauberian theorems [10] we immediately have

$$P_0(t) \sim t^{-1/2(1+\ln r/\ln 2)} \tag{10}$$

which, according to (3), finally gives

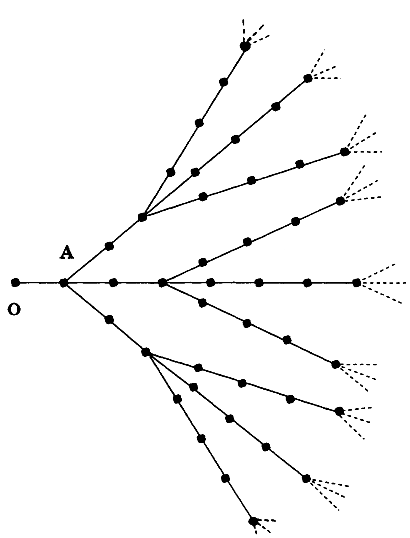

FIG. 1. An NT_D with $r=3$ ($D = d_f = \tilde{d} = 1 + \log r/\log 2$).

$$\tilde{d}=1+\frac{\ln r}{\ln 2} \tag{11}$$

that coincides with d_f.

This completes our calculation and gives the proof of existence of fractals with dimensional degeneracy and consequent nonanomalous diffusion. Since the fractal dimension of these trees can be chosen to be arbitrarily large fixing an appropriate value of r, these fractals also have an arbitrarily large spectral dimension, violating the hypothesis of the generalized Mermin-Wagner theorem [4] and therefore providing a very interesting testing ground for new ideas and theories about phase transitions and critical behavior on fractals. A detailed study of statistical models on NT_D and other tree structures will be the subject of a forthcoming paper [11].

[1] S. Alexander and R. Orbach, J. Phys. Lett. **43**, L625 (1982).
[2] See also D. Dhar, J. Math. Phys. **18**, 577 (1977).
[3] K. Hattori, T. Hattori, and H. Watanabe, Prog. Theor. Phys. Suppl. **92**, 108 (1987).
[4] D. Cassi, Phys. Rev. Lett. **68**, 3631 (1992).
[5] P. G. Doyle and J. L. Snell, *Random Walks and Electric Networks* (Mathematical Association of America, Oberlin, OH, 1984).
[6] R. Burioni and D. Cassi (unpublished).
[7] M. Suzuki, Prog. Theor. Phys. **69**, 65 (1983).
[8] D. Cassi and S. Regina, Phys. Rev. Lett. **70**, 1647 (1993).
[9] V. A. Kaimanovich, Potential Analysis **1**, 61 (1992).
[10] G. H. Hardy, *Divergent Series* (Oxford University Press, New York, 1949).
[11] R. Burioni and D. Cassi, Università di Parma–Fisica Report No. UPRF-93-374 (unpublished).